DIFFERENTIAL, DIFFERENCE, AND INTEGRAL METHODS OF MEASURING
TEMPERATURE DEPENDENCE OF COEFFICIENT OF THERMAL CONDUCTIVITY

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A "temperature-step method" is proposed for the measurement of temperature distribution along a sample from which a differential coefficient of thermal conductivity is determined. The physical meaning of the difference coefficient of thermal conductivity as an average value for a given temperature range is emphasized. The possibility of making exact measurements of an integral coefficient of thermal conductivity is justified.

Determination of the coefficient of thermal conductivity is based on the Fourier equation representing the relation between the thermal flux density q and the temperature gradient grad T,

$$q = - K \operatorname{grad} T. \tag{1}$$

We consider stationary methods for the determination of the coefficient of thermal conductivity K, which is the proportionality factor in Eq. (1). Experimental stationary methods for the determination of K may be based either on direct determination of the physical quantities appearing in Eq. (1) or on certain variations of the experimental methods of measurement required by transformations of Eq. (1).

The aim of this paper is a comparison and discussion of the experimental conditions needed for making measurements to determine the coefficient of thermal conductivity by the three methods mentioned in the title.

Differential Method for Measuring the Temperature Dependence of the Coefficient of Thermal Conductivity. The Fourier equation (1) for the one-dimensional case

$$q = - K_{\mathrm{d}}(T) \frac{dT}{dx}. \tag{2}$$

is the basis for our discussion of the differential method of measurement.

It is necessary to determine the local value of the temperature gradient dT/dx in order to calculate the coefficient of thermal conductivity $K_d(T)$. At the present time, there are no instruments for measuring local values of the temperature gradient. If we know the temperature distribution along a specimen, we can determine the local value of the temperature gradient by analytic or graphic methods. The temperature distribution $T(x)$ can be measured by means of a series of thermometers which are in thermal contact with the specimen or by means of only one thermometer which slides along the specimen.

In both cases, a specimen with a sufficient degree of uniformity is required, and one with a length which is determined by the number of points at which we intend to measure the temperature. There is no sense in reducing the spacing between thermometers

to less than 1 cm if only because the location uncertainty of the thermal attachment of a thermometer is of the order of 1 mm.

We propose a method for the determination of the temperature distribution along a uniform sample by means of only two thermometers. This method is very suitable in the low-temperature region and its advantage is that it can be used for all specimens ordinarily tested. A necessary condition for the application of the proposed method of "successive temperature steps" is the capability of controlling in a given manner the temperature of the cold block to which the test specimen is attached.

We first establish the thermal-flux density flowing along the specimen.

When the temperature of a copper block is $_1T_{bl}$, let a thermometer located near the heater indicate a temperature T_2 and a thermometer near the cold block indicate a temperature T_1. The next step is adjustment of the temperature of the copper block to $_2T_{bl}$ (for example, by an increase in temperature) in such a way that the thermometer near the cold block (previously indicating a temperature T_1) reaches the temperature T_2 (previously indicated by the thermometer near the heater). Now the thermometer near the heater reaches the temperature T_3. Each succeeding n-th temperature step will consist of a selection of the temperature $_nT_{bl}$ of the copper block such that the thermometer near the cold block reaches the temperature T_n (which the thermometer at the heater had indicated previously) and the thermometer near the heater will now indicate a new temperature T_{n+1}, etc. One can also change the temperature of the copper block by reducing the temperature if there are experimental capabilities to do this.

Using the method of "successive temperature steps" given above, one can determine the temperature distribution along a uniform hypothetical specimen with a length $n\Delta x$ (where n is the number of temperature steps and Δx is the spacing between the two thermometers). Thus, the differential method for the determination of $K_d(T)$ becomes completely achievable in an actual experimental device for low-temperature studies of thermal conductivity. The method of temperature steps has already been used [1].

Difference Method for Measuring the Temperature Dependence of the Coefficient of Thermal Conductivity. The Fourier equation written in the form of the difference equation

$$q = -\overline{K}_p(T_1,\ T_2)\frac{T_2 - T_1}{\Delta x}. \tag{3}$$

is the basis for the most widely used difference method for the determination of the coefficient of thermal conductivity. Since the coefficient of thermal conductivity generally depends on temperature, the proportionality factor $\overline{K}_p$ is some average value [2, 3] in the temperature range T_2-T_1.

The stronger the temperature dependence of the thermal conductivity of the test specimen, the smaller need be the temperature difference $\Delta T = T_2 - T_1$ between thermometers. The accuracy in determination of temperature is of the order of $10^{-3}°K$ for the best low-temperature thermometers so that the ΔT which is maintained between thermometers cannot be less than $10^{-1}°K$ in order that the error in the determination of ΔT not exceed a few percent. If the test specimen is characterized by a very steep dependence $K(T)$, it can turn out that the coefficient of thermal conductivity determined by the difference method will differ greatly from the values of the thermal conductivities $K(T)$ and $K(T \pm \Delta T_1)$ at the temperature points even for a temperature range $\Delta T_1 = 10^{-1}°K$.

The percentage error in the determination of Δx increases as the distance between thermometers decreases; this distance should not be less than 10 mm.

The average temperature gradient $\Delta T/\Delta x$ appearing in Eq. (3) can be equated to the local value of the gradient dT/dx appearing in Eq. (2) only for the case of linear temperature distribution along the specimen.

For the general case of nonlinear temperature distribution, one can formally write

$$\lim_{\substack{\Delta T \to 0 \\ \Delta x \to 0}} \frac{\Delta T}{\Delta x} \to \frac{dT}{dx}. \tag{4}$$

However, condition (4) is unrealizable in practice. Under actual experimental conditions we have

$$\lim_{\substack{\Delta T \to \Delta T\text{min} \\ \Delta x = \Delta x_0 = \text{const}}} \frac{\Delta T}{\Delta x} \to \frac{\Delta T_{\min}}{\Delta x_0} \neq \frac{dT}{dx}, \tag{5}$$

where $\Delta T_{\min}$ is of the order of $0.1°$K and Δx_0 is of the order of 1 cm.

Because of Eq. (5), we always have

$$\lim_{\Delta T \to \Delta T_{\min}} \bar{K}_{\text{p}}(T + \Delta T) \neq K_{\text{d}}(T). \tag{6}$$

The discrepancy between $\bar{K}_{\text{p}}(T + \Delta T)$ and $K_{\text{d}}(T)$ for uniform specimens will be more marked the greater the value of dK/dT in the temperature range ΔT.

Knowing the relation $K_{\text{d}}(T)$ and the analytic expression for $\bar{K}_{\text{p}}(T + \Delta T)$, one can express $\bar{K}_{\text{p}}(K_{\text{d}})$ mathematically. The physical significance of the gradient $\Delta T/\Delta x$ appearing in Eq. (3) is that we formally replace the actual nonlinear temperature distribution by a linear distribution. All the consequences of this method have an effect on the physical content of the difference coefficient of thermal conductivity, $\bar{K}_{\text{p}}(T + \Delta T)$, defined by Eq. (3).

So-Called Integral Method for Measuring the Temperature Dependence of the Coefficient of Thermal Conductivity. By modifying the technique of the difference method for measuring the coefficient of thermal conductivity, one can create conditions for the realization of the so-called integral method of measuring the coefficient of thermal conductivity. For this purpose, it is necessary to maintain a constant temperature at thermometer 2 (which is located close to the cold block), for example, and to measure the change in temperature at thermometer 1 (near the heater) as a function of the change in thermal flux density flowing along the specimen. The constancy of the temperature at thermometer 2 can be achieved by controlling the temperature of the cold block in a given manner. An expression for the coefficient of thermal conductivity determined by the integral method can be obtained in the following manner (noting only some comments in [4] and [5]):

$$q = - K(T) \frac{dT}{dx}. \tag{7}$$

Separating variables and integrating, we have

$$\int_0^l q \, dx = - \int_{T_0 = \text{const}}^T K(T) \, dT. \tag{8}$$

From Eq. (8) we obtain

$$ql = - [\varkappa(T) - \varkappa(T_0)], \tag{9}$$

where $\varkappa$ is a function satisfying the condition

$$d\varkappa = K(T) \, dT. \tag{10}$$

Differentiating both sides of Eq. (9), we have

$$l \, dq = - K_{\text{i}}(T) \, dT, \tag{11}$$

whence

$$K_i(T) = -l \frac{dq}{dT}.$$

(12)

$K_i(T)$ is the coefficient of thermal conductivity determined by the integral method which is related to a given temperature T.

For measurements of $K_i(T)$, it is necessary to determine the change in temperature at thermometer 1 as a function of the change in the thermal-flux density q. The temperature at thermometer 2 must be maintained constant during this time; this can be accomplished by means of a device for temperature control and stabilization.

We take the temperature at thermometer 1 under stationary conditions.

Comparing Eqs. (2) and (12), we see that the coefficients $K_d(T)$ and $K_i(T)$ are defined for a given, rigorously established temperature T while the coefficient $\bar{K}_p$ is defined by Eq. (3) in the temperature range T + ΔT. It therefore makes sense to ask how well experimentally determined values of $K_d(T)$ and $K_i(T)$ agree.

At the present time, there are possibilities for making precise measurements of the coefficient of thermal conductivity by the differential and integral methods. Thus far, such comparison measurements have not been made. We have used the "temperature-step method" to determine the thermal conductivity $K_d(T)$ of an Ag test specimen [1].

Detailed methodological analysis of the integral method for measuring thermal conductivity and its comparison with the differential method will be the subject of a separate publication.

Comparison of the coefficient of thermal conductivity $\bar{K}_p$ determined by the difference method with the coefficient of thermal conductivity K_d determined by the differential method is only possible in the case of a known analytic relation between $\bar{K}_p$ and K_d.

Some remarks about the possibilities of determining $\bar{K}_p$ have been made [6].

Using the temperature-step method, we can measure the temperature distribution T(x) with sufficient accuracy for a given stationary thermal-flux density q. If we know the distribution T(x), we can determine the coefficient of thermal conductivity $K_d(T)$ by the differential method on the basis of Eq. (2).

In the most common difference method for the determination of the coefficient of thermal conductivity $\bar{K}_p$, two thermometers are used at a spacing of Δx to determine the temperature difference $T_2 - T_1$. Equation (3) is the basis for the determination of the coefficient $\bar{K}_p$. In principle, the coefficient $\bar{K}_p(T + \Delta T)$ can never agree with the coefficient $K_d(T)$ and this disagreement is greater the steeper the temperature dependence of thermal conductivity.

The so-called integral method for the determination of the coefficient of thermal conductivity $K_i(T)$ [Eq. (12)] is based on certain experimental modifications of the preceding methods. In this case, one must maintain a constant temperature at one of the thermometers and determine the change in temperature at the other thermometer as a function of the stationary thermal-flux density flowing along the specimen.

It appears advisable to make an exact experimental comparison of the coefficients $K_d(T)$ and $K_i(T)$ measured by the differential and integral methods in the region of strong temperature dependence of thermal conductivity.

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NOTATION

q is the heat-flux density; grad T is the temperature gradient; K is the thermal conductivity; K_d is the thermal conductivity predicted by using differential methods; dT/dx is the local temperature gradient value; T(x) is the temperature distribution; $_iT_{bl}$ (i = 1, 2, ..., n) is the heat sink temperature; T_i is the temperature of thermometers on the sample; n is the number of temperature steps; Δx is the distance between

thermometers; $K_p(T + \Delta T)$ is the thermal conductivity determined by the difference method; $\Delta T/\Delta x$ is the mean-temperature gradient; l is the distance of the thermometer from isothermal surface normal to the heat flux; T_0 is the constant temperature of isothermal surface; $K_i(T)$ is the thermal conductivity determined by the integral method.

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